Title	The regularized mean curvature flow for invariant hypersurfaces in a Hilbert space with an almost free Lie group action (Development of group actions and submanifold theory)
Author(s)	小池, 直之
Citation	数理解析研究所講究録 (2014), 1929: 75-90
Issue Date	2014-12
URL	http://hdl.handle.net/2433/223540
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録
第 1929 巻 2014 年 75-90

The regularized mean curvature flow for invariant hypersurfaces in a Hilbert space with an almost free Lie group action

Naoyuki Koike
Department of Mathematics, Faculty of Science
Tokyo University of Science

1 Introduction

From 1984、Huisken and many geometers studied the mean curvature flow for a hypersurface or a submanifold (of higher codimension) in a Euclidean space as an evolution of the immersion. The existenceness (and the uniqueness) of the mean curvature flow for an initial hypersurface (or submanifold) $f : M \hookrightarrow \mathbb{R}^m$ in short time is assured under the assumption of the compactness of M or under the assumptions of the invariantness of $f(M)$ by some Lie group action consisting of isometries of $\mathbb{R}^m$ and the compactness of the $f(M)/G$.

The study of the mean curvature flow for a submanifold M in a (general) Riemannian manifold $\widetilde{M}$ also has been done by many geometers. The evolutions of various geometric quantities (tensor fields) along the mean curvature flow are obtained by calculating the evolutions of their components with respect to local coordinates of M and $\widetilde{M}$. In particular, in the case where $\widetilde{M}$ is an Euclidean space, it is simpler to treat because we can use the fact that $\widetilde{M}$ is a linear space.

In order to define and study the mean curvature flow for an infinite dimensional submanifold $f : M \hookrightarrow V$ in a Hilbert space V, we must first define the mean curvature vector of f. The mean curvature vector of f should be defined by using the traces of the shape operators of f for unit normal vectors but how the trace should be defined is important problem. The geodesic closed ball with respect to the induced metric on the submanifold is not compact. Hence in order to assure the existenceness (and the uniqueness) of the mean curvature flow for an initial submanifold $f : M \hookrightarrow V$ in short time, we must impose the conditions of the invariantness of $f(M)$ by some infinite dimensional Lie group action consisting of isometries of V and the compactness of the $f(M)/G$. Also, since M is a Hilbert manifold, we cannot use a local coordinate. Hence we shall calculate the evolutions of various geometric quantities by using technique in the theory of the vector bundle.

Under the above background, we studied the regularized mean curvature flow for a G-invariant regularizable hypersurface in a Hilbert space V equipped with an almost free Hilbert Lie group isometric action $G \curvearrowright V$ whose orbits are minimal. See Sections 2 and 3 about the definitions of the regularizable hypersurface and the regularized mean curvature flow. This study can be applied to the study of the mean curvature flow in the orbit space V/G (which is a Riemannian orbifold).

2 Regularizable submanifolds

In this section, we shall state the definition of a regularizable submanifolds in a (separable) Hilbert space. Let V be a (separable) Hilbert space, M be a Hilbert manifold and f be an immersion of M into V. Denote by $T^\perp M$, A and $\exp^\perp$ the normal bundle, the shape tensor and the normal exponential map of f, respectively. If the following three conditions hold, then $f : M \hookrightarrow V$ is called a *proper Fredholm submanifold*:

(i) $\operatorname{codim} f(M) < \infty$,
(ii) the restriction of $\exp^\perp$ to the unit normal ball bundle of f is proper,
(iii) the differential of $\exp^\perp$ at each point of $T^\perp M$ is a Fredholm operator.

Note that the shape operators A_v ($v \in T^\perp M$) of a proper Fredholm submanifold are compact operators. This notion was introduced by C. L. Terng ([Te]) in 1989. In 2006, E. Heintze, X. Liu and C. Olmos ([HLO]) defined the regularized trace $\mathrm{Tr}_r\, A_v$ of the shape operator A_v as follows:

$$\mathrm{Tr}_r\, A_v := \sum_{i=1}^{\infty}(\mu_i^+ + \mu_i^-)$$
$$(\mu_1^- \leq \mu_2^- \leq \cdots \leq 0 \leq \cdots \leq \mu_2^+ \leq \mu_1^+ \ : \text{ the spectrum of } A_v)$$

Assume that $f : M \hookrightarrow V$ is proper Fredholm. Furthermore, if there exist the regularized trace of A_v and the (usual) trace of A_v^2 for any unit normal vector v of f, then $f : M \hookrightarrow V$ is called a *regularizable submanifold*. This notion was introduced by E. Heintze, X. Liu and C. Olmos ([HLO]). Let $f : M \hookrightarrow V$ be a regularizable submanifold. The *regularized mean curvature vector* of f is defined as the normal vector field H of f satisfying

$$\langle H, v\rangle = \mathrm{Tr}_r\, A_v \ (\forall v \in T^\perp M),$$

where $\langle\ ,\ \rangle$is the inner product of V. The norm of H is called the *regularized mean curvature* of f. In particular, if $H = 0$, then $f : M \hookrightarrow V$ is said to be *minimal*. On the other hand, the regularized Laplacian $\triangle_r f$ of the vector-valued function f is defined by

$$\langle \triangle_r f, v\rangle = \mathrm{Tr}_r \langle (\nabla df)(\cdot,\cdot), v\rangle^\sharp \quad (\forall v \in T^\perp M),$$

where ∇ is the Riemannian connection of the induced metric g on M by f and $\langle (\nabla df)(\cdot,\cdot), v\rangle^\sharp$ is the $(1,1)$-tensor field on M defined by $g_t(\langle (\nabla df)(\cdot,\cdot), v\rangle^\sharp(X), Y) = \langle (\nabla df)(X,Y), v\rangle$ $(X, Y \in TM)$ It is easy to show that $\triangle_r f = H$ holds.

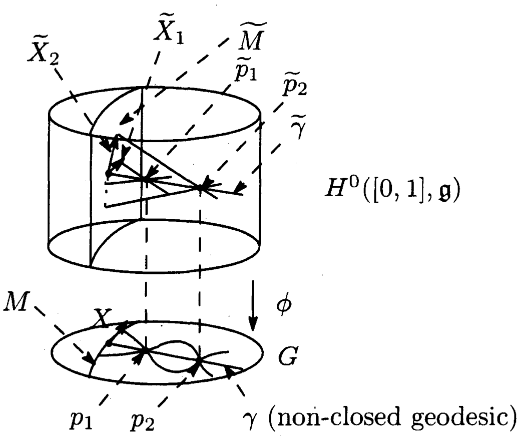
Example 2.1. Let G be a compact semi-simple Lie group equipped with a bi-invariant metric and $M(\subset G)$ be an embedded submanifold in G. The *parallel transport map* $\phi : H^0([0,1], \mathfrak{g}) \to G$ for G is definde by

$$\phi(u) := g_u(1) \quad (u \in H^0([0,1], \mathfrak{g}))$$
$$(\ g_u \in H^1([0,1], G) \ \text{ s.t. } \text{“}g_u(0) = e,\ (R_{g_u(t)})_*^{-1}(g_u'(t)) = u(t)\ (\forall t \in [0,1])\text{”}\ ),$$

where $H^0([0,1], \mathfrak{g})$ is the (separable) Hilbert space of all H^0-paths in the Lie algebra $\mathfrak{g}$ of G and $H^1([0,1], G)$ is the Hilbert Lie group of all H^1-paths in G. Then it is shown that $\widetilde{M} := \phi^{-1}(M)$ is a regularizable submanifold in $H^0([0,1], \mathfrak{g})$ (see [HLO]). The relation between the focal structures of M and $\widetilde{M}$ is as in Figure 1. In the case where M is curvature-adapted (i.e., $R(v)(T_xM) \subset T_xM$, $[A_v, R(v)] = 0$ ($\forall x \in M$, $\forall v \in T_x^\perp M$)), we shall state the relation between the spectrums of the shape operators of M and $\widetilde{M}$, where R is the curvature tensor of G and $R(v)$ is the normal Jacobi operator for v (i.e., $R(v) := R(\cdot, v)v$). Take a unit normal vector v of M at $x \in M$. Let v_u^L be the horizontal lift of v to $u \in \phi^{-1}(x)$. Denote by A and $\widetilde{A}$ the shape operators of M and $\widetilde{M}$, respectively. Set $D_\lambda^A := \mathrm{Ker}(A_v - \lambda\,\mathrm{id})$ ($\lambda \in \mathrm{Spec}\,A_v$) and $D_\mu^R := \mathrm{Ker}(R(v) - \mu\,\mathrm{id})$ ($\mu \in \mathrm{Spec}\,R(v)$). Then $\mathrm{Spec}\,\widetilde{A}_{v_u^L} \setminus \{0\}$ is described as

$$\begin{aligned}
&\mathrm{Spec}\,\widetilde{A}_{v_u^L} \setminus \{0\} \\
&= \{\lambda \mid \lambda \in \mathrm{Spec}\,A_v \text{ s.t. } D_\lambda^A \cap D_0^R \neq \{0\}\} \\
&\quad \bigcup \left\{ \frac{\mu}{\arctan(\mu/\lambda) + j\pi} \,\middle|\, (\lambda, \mu) \in \mathrm{Spec}\,A_v \times \mathrm{Spec}\,R(v) \text{ s.t. } D_\lambda^A \cap D_\mu^R \neq \{0\},\ j \in \mathbb{Z} \right\} \\
&\quad \bigcup \left\{ \frac{\mu}{j\pi} \,\middle|\, \mu \in \mathrm{Spec}\,R(v) \text{ s.t. } D_\mu^R \cap T_x^\perp M \neq \{0\},\ j \in \mathbb{Z} \setminus \{0\} \right\}.
\end{aligned}$$

From this description, it follows that the regularized trace of $\widetilde{A}_{v_u^L}$ exists. Also, it follows that the regularized mean curvature vector of $\widetilde{M}$ is the horizontal lift of the mean curvature vector of M.



$X \in$ "the nullity space of focal points p_i ($i \in \mathbb{N}$)"
$\widetilde{X}_i$ ($i \in \mathbb{N}$) $\in$ "the nullity space of a focal point $\widetilde{p}_i$
($\widetilde{X}_i$ ($i \in \mathbb{N}$) are linearly independent.)

Figure 1.

3 Regularized mean curvature flow

In this section, we shall state the definition of a regularized mean curvature flow in a (separable) Hilbert space. Let V and M be as in the previous section. Let f_t $(0 \le t < T)$ be a C^∞-family of regularizable immersions of M into V. Denote by H_t the regularized mean curvature vector of f_t. Define a map $F : M \times [0, T) \to V$ by $F(x,t) = f_t(x)$ $((x,t) \in M \times [0,T))$. We call f_t's $(0 \le t < T)$ the *regularized mean curvature flow* if the following evolution equation holds:

$$\frac{\partial F}{\partial t} = \triangle_t^r f_t, \tag{2.1}$$

where $\triangle_t^r f_t$ is the regularized Laplacian of f_t (i.e., $\triangle_t^r f_t = H_t$). In general, the existenceness and the uniqueness (in short time) of solutions of this evolution equation satisfying any initial condition has not been shown yet. For we cannot apply the Hamilton's result ([Ha]) to this evolution equation because it is regarded as the evolution equation for sections of the *infinite* dimensional vector bundle $M \times V$ over M. However we can show the existenceness and the uniqueness (in short time) of solutions of this evolution equation in the following special case. We consider the case where V equips an almost free and isometric Hilbert Lie group action $G \curvearrowright V$ with minimal regulariazable fibres and where $f : M \hookrightarrow V$ is a G-invariant embedded hypersurface in V such that $f(M)/G$ is compact. Then it is shown that the rgularized mean curvature flow for f uniquely exists in short time.

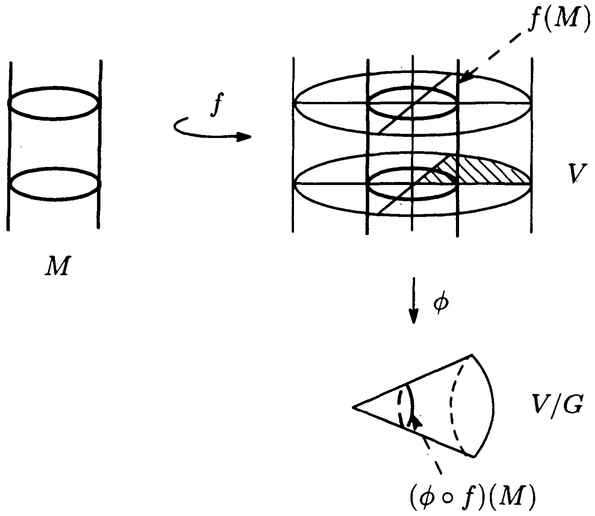


Figure 2.

Example 3.1. Let G be a compact semi-simple Lie group equipped with a bi-invariant metric and K be a closed subgroup of G. Also, let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively. Assume that $(\mathfrak{g}, \mathfrak{k})$ admits a reductive decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Also, let Γ be a discrete subgroup of G. We define a Hilbert Lie group $P(G, \Gamma \times K)$ by

$$P(G, \Gamma \times K) := \{g \in H^1([0,1], G) \mid (g(0), g(1)) \in \Gamma \times K\}.$$

This group $P(G, \Gamma \times K)$ acts on $H^0([0,1], \mathfrak{g})$ as the action of a Gauge action on the space of the connections, where $H^1([0,1], G)$ is the Hilbert Lie group of all H^1-paths in G. This action is an almost free and isometric action whose orbits are minimal regularizable submanifolds and the orbits space $H^0([0,1], \mathfrak{g})/P(G, \Gamma \times K)$ is equal to $\Gamma \backslash G/K$.

4 The mean curvature flow for suborbifolds

In this section, we shall define the notion of the mean curvaure flow for a suborbifold in a Riemannian orbifold. First we recall the notions of a Riemannian orbifold and a suborbifold following to [AK,GKP,Sh,Th]. Let M be a paracompact Hausdorff space and $(U, \phi, \widetilde{U}/\Gamma)$ a triple satisfying the following conditions:

(i) U is an open set of M,
(ii) $\widehat{U}$ is an open set of $\mathbb{R}^n$ and Γ is a finite subgroup of the C^k-diffeomorphism group $\mathrm{Diff}^k(\widehat{U})$ of $\widehat{U}$,
(iii) ϕ is a homeomorphism of U onto $\widehat{U}/\Gamma$.

Such a triple $(U, \phi, \widehat{U}/\Gamma)$ is called an *n-dimensional orbifold chart*. Let $\mathcal{O} := \{(U_\lambda, \phi_\lambda, \widehat{U}/\Gamma_\lambda) \mid \lambda \in \Lambda\}$ be a family of n-dimensional orbifold charts of M satisfying the following conditions:

(O1) $\{U_\lambda \mid \lambda \in \Lambda\}$ is an open covering of M,
(O2) For any $\lambda, \mu \in \Lambda$ with $U_\lambda \cap U_\mu \neq \emptyset$ and any $x \in U_\lambda \cap U_\mu$, there exists an n-dimensional orbifold chart $(W, \psi, \widehat{W}/\Gamma')$ such that C^k-embeddings $\rho_\lambda : \widehat{W} \hookrightarrow \widehat{U}_\lambda$ and $\rho_\mu : \widehat{W} \hookrightarrow \widehat{U}_\mu$ satisfying $\phi_\lambda^{-1} \circ \pi_{\Gamma_\lambda} \circ \rho_\lambda = \psi^{-1} \circ \pi_{\Gamma'}$ and $\phi_\mu^{-1} \circ \pi_{\Gamma_\mu} \circ \rho_\mu = \psi^{-1} \circ \pi_{\Gamma'}$, where π_{Γ_λ}, π_{Γ_μ} and $\pi_{\Gamma'}$ are the orbit maps of Γ_λ, Γ_μ and Γ', respectively.

Such a family $\mathcal{O}$ is called an *n-dimensional C^k-orbifold atlas* of M and the pair $(M, \mathcal{O})$ is called an *n-dimensional C^k-orbifold*. Let $(U_\lambda, \phi_\lambda, \widehat{U}_\lambda/\Gamma_\lambda)$ be an n-dimensional orbifold chart around $x \in M$. Then the group $(\Gamma_\lambda)_{\widehat{x}} := \{b \in \Gamma_\lambda \mid b(\widehat{x}) = \widehat{x}\}$ is unique for x up to the conjugation, where $\widehat{x}$ is a point of $\widehat{U}_\lambda$ with $(\phi_\lambda^{-1} \circ \pi_{\Gamma_\lambda})(\widehat{x}) = x$. Denote by $(\Gamma_\lambda)_x$ the conjugate class of this group $(\Gamma_\lambda)_{\widehat{x}}$, This conjugate class is called the *local group at x*. If the local group at x is not trivial, then x is called a *singular point* of $(M, \mathcal{O})$. Denote by $\mathrm{Sing}(M, \mathcal{O})$ (or $\mathrm{Sing}(M)$) the set of all singular points of $(M, \mathcal{O})$. This set $\mathrm{Sing}(M, \mathcal{O})$ is called the *singular set* of $(M, \mathcal{O})$.

Let $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ be orbifolds, and f a map from M to N. If, for each $x \in M$ and each pair of an orbifold chart $(U_\lambda, \phi_\lambda, \widehat{U}_\lambda/\Gamma_\lambda)$ of $(M, \mathcal{O}_M)$ around x and an orbifold chart $(V_\mu, \psi_\mu, \widehat{V}_\mu/\Gamma'_\mu)$ of $(N, \mathcal{O}_N)$ around $f(x)$ $(f(U_\lambda) \subset V_\mu)$, there exists a C^k-map $\widehat{f}_{\lambda,\mu} : \widehat{U}_\lambda \to \widehat{V}_\mu$ with $f \circ \phi_\lambda^{-1} \circ \pi_{\Gamma_\lambda} = \psi_\mu^{-1} \circ \pi_{\Gamma'_\mu} \circ \widehat{f}_{\lambda,\mu}$, then f is called a *C^k-orbimap* (or simply a *C^k-map*). Also $\widehat{f}_{\lambda,\mu}$ is called a *local lift* of f with respect to $(U_\lambda, \phi_\lambda, \widehat{U}_\lambda/\Gamma_\lambda)$ and $(V_\mu, \psi_\mu, \widehat{V}_\mu/\Gamma'_\mu)$. Furthermore, if each local lift $\widehat{f}_{\lambda,\mu}$ is an immersion, then f is called a *C^k-orbiimmersion* (or simply a *C^k-immersion*) and $(M, \mathcal{O}_M)$ is called a *C^k-(immersed) suborbifold* in $(N, \mathcal{O}_N, g)$. Similarly, if each local lift $\widehat{f}_{\lambda,\mu}$ is a submersion, then f is called a *C^k-orbisubmersion*.

Now we shall define the notion of the mean curvature flow for a C^∞-suborbifold in a C^∞-Riemannian orbifold. Let f_t $(0 \leq t < T)$ be a C^∞-family of C^∞-orbiimmersions of a C^∞-orbifold $(M, \mathcal{O}_M)$ into a C^∞-Riemannian orbifold $(N, \mathcal{O}_N, g)$. Assume that, for each $(x_0, t_0) \in M \times [0, T)$ and each pair of an orbifold chart $(U_\lambda, \phi_\lambda, \widehat{U}_\lambda/\Gamma_\lambda)$ of $(M, \mathcal{O}_M)$ around x_0 and an orbifold chart $(V_\mu, \phi_\mu, \widehat{V}_\mu/\Gamma'_\mu)$ of $(N, \mathcal{O}_N)$ around $f_{t_0}(x_0)$ such that $f_t(U_\lambda) \subset V_\mu$ for any $t \in [t_0, t_0 + \varepsilon)$ (ε : a sufficiently small positive number), there exists local lifts $(\widehat{f}_t)_{\lambda,\mu} : \widehat{U}_\lambda \to \widehat{V}_\mu$ of f_t $(t \in [t_0, t_0 + \varepsilon))$

such that they give the mean curvature flow in $(\widehat{V}_\mu, \widehat{g}_\mu)$, where $\widehat{g}_\mu$ is the local lift of g to $\widehat{V}_\mu$. Then we call f_t $(0 \le t < T)$ the *mean curvature flow* in $(N, \mathcal{O}_N, g)$.

Theorem 4.1([K3]). *For any C^∞-orbiimmersion f of a compact C^∞-orbifold into a C^∞-Riemannian orbifold, the mean curvature flow starting from f exists uniquely in short time.*

Proof. Let f be a C^∞-orbiimmersion of an n-dimensional compact C^∞-orbifold $(M, \mathcal{O}_M)$ into an $(n+r)$-dimensional C^∞-Riemmannian orbifold $(N, \mathcal{O}_N, g)$. Fix $x_0 \in M$. Take an orbifold chart $(U_\lambda, \phi_\lambda, \widehat{U}_\lambda/\Gamma_\lambda)$ of $(M, \mathcal{O}_M)$ around x_0 and an orbifold chart $(V_\mu, \psi_\mu, \widehat{V}_\mu/\Gamma'_\mu)$ of $(N, \mathcal{O}_N)$ around $f(x_0)$ such that $f(U_\lambda) \subset V_\mu$ and that $\widehat{U}_\lambda$ is relative compact. Also, let $\widehat{f}_{\lambda,\mu} : \widehat{U}_\lambda \hookrightarrow \widehat{V}_\mu$ be a local lift of f and $\widehat{g}_\mu$ a local lift of g (to $\widehat{V}_\mu$). Since $\widehat{U}_\lambda$ is relative compact, there exists the mean curvature flow $(\widehat{f}_{\lambda,\mu})_t : \widehat{U}_\lambda \hookrightarrow (\widehat{V}_\mu, \widehat{g}_\mu)$ $(0 \le t < T)$ starting from $\widehat{f}_{\lambda,\mu} : \widehat{U}_\lambda \hookrightarrow (\widehat{V}_\mu, \widehat{g}_\mu)$. Since $\widehat{f}_{\lambda,\mu}$ is projetable to $f|_{U_\lambda}$ and $\widehat{g}_\mu$ is Γ'_μ-invariant, $(\widehat{f}_{\lambda,\mu})_t$ $(0 \le t < T)$ also are projectable to maps of U_λ into V_μ. Denote by $(f_{\lambda,\mu})_t$'s these maps of U_λ into V_μ. It is clear that $(f_{\lambda,\mu})_t$ $(0 \le t < T)$ is the mean curvature flow starting from $f|_{U_\lambda}$. Hence, it follows from the arbitrariness of x_0 and the compactness of M that the mean curvature flow starting from f exists uniquely in short time. q.e.d.

例 4.1.

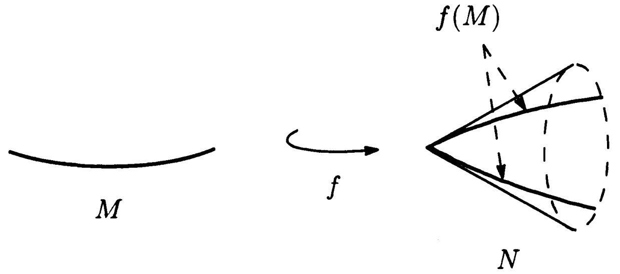


Figure 3.

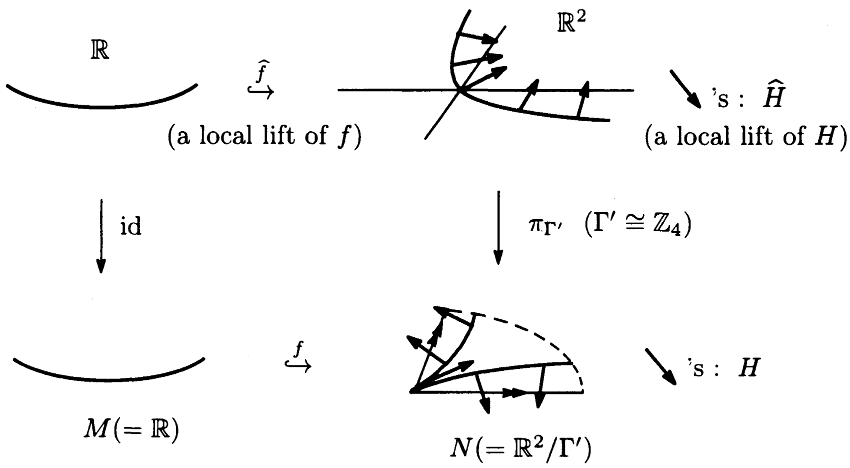


Figure 4.

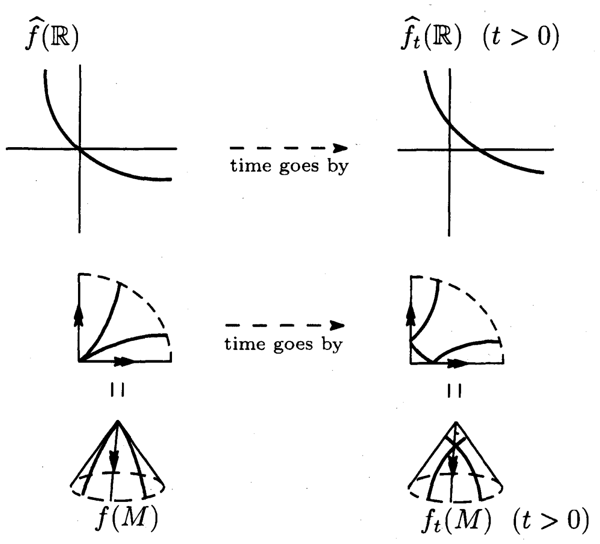


Figure 5.

例 4.2.

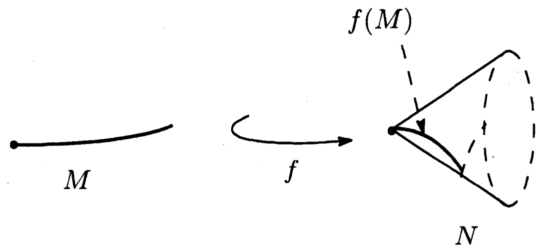


Figure 6.

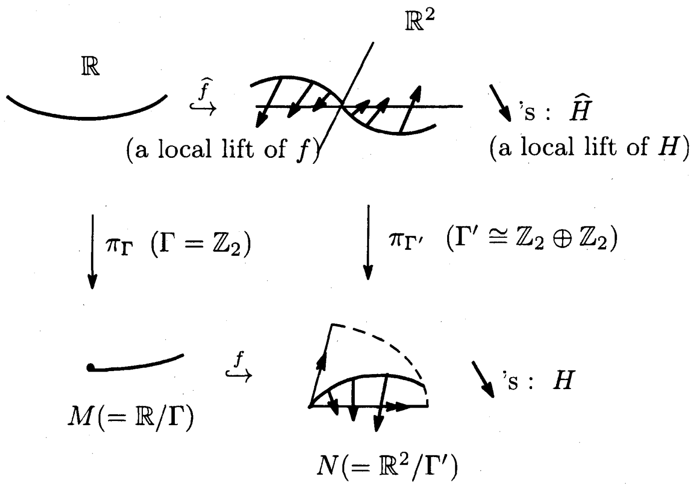


Figure 7.

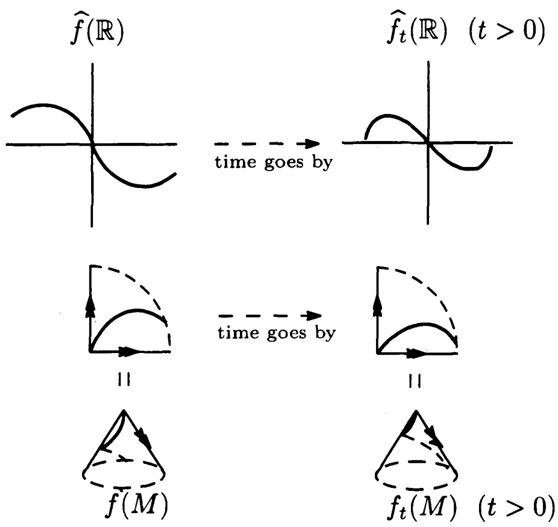


Figure 8.

5 Evolution equations

Let $G \curvearrowright V$ be an isometric almost free action with minimal regularizable orbit of a Hilbert Lie group G on a Hilbert space V equipped with an inner product $\langle\ ,\ \rangle$. The orbit space V/G is a (finite dimensional) C^∞-orbifold. Let $\phi : V \to V/G$ be the orbit map and set $N := V/G$. Give N the Riemannian orbimetric such that ϕ is a Riemannian orbisubmersion. Let $f : M \hookrightarrow V$ be a G-invariant submanifold such that $(\phi \circ f)(M)$ is compact. For this immersion f, we can take an orbiimmesion $\overline{f}$ of a compact orbifold $\overline{M}$ into N and an orbisubmersion $\phi_M : M \to \overline{M}$ with $\phi \circ f = \overline{f} \circ \phi_M$. Let $\overline{f}_t$ $(0 \le t < T)$ be the mean curvature flow for $\overline{f}$. The existenceness and the uniqueness of this flow in short time is assured by Theorem 4.1. Define a map $\overline{F} : \overline{M} \times [0,T) \to N$ by $\overline{F}(x,t) := \overline{f}_t(x)$ $((x,t) \in \overline{M} \times [0,T))$. Denote by H the regularized mean curvature vector of f and $\overline{H}$ the mean curvature vector of $\overline{f}$. Since ϕ has minimal regularizable fibres, H is the horizontal lift of $\overline{H}$. Take $x \in \overline{M}$ and $u \in \phi_M^{-1}(x)$. Define a curve $c_x : [0,T) \to N$ by $c_x(t) := \overline{f}_t(x)$ and let $(c_x)_u^L : [0,T) \to V$ be the horizontal lift of c_x for $f(u)$. Define an immersion $f_t : M \hookrightarrow V$ by $f_t(u) = (c_x)_u^L(t)$ $(u \in \widetilde{M})$ and a map $F : M \times [0,T) \to V$ by $F(u,t) = f_t(u)$ $((u,t) \in M \times [0,T))$.

Proposition 5.1([K3]). *The flow f_t $(0 \le t < T)$ is the regularized mean curvature flow for f.*

Proof. Denote by $\overline{H}_t$ the mean curvature vector of $\overline{f}_t$ and H_t the regularized mean curvature vector of f_t. Take any $(u,t) \in M \times [0,T)$. Set $x := \phi_M(u)$. It is clear that $\phi \circ f_t = \overline{f}_t \circ \phi_M$. Hence, since each fibre of ϕ is regularizable and minimal, $(H_t)_u$ coincides with one of the horizontal lifts of $(\overline{H}_t)_x$ to $f_t(u)$. On the other hand, from the definition of F, we have $\frac{\partial F}{\partial t}(u,t) = ((c_x)_u^L)'(t)$, which is one of the horizontal lifts of $(\overline{H}_t)_x$ to $f_t(u)$. These facts together with $\frac{\partial F}{\partial t}(u,0) = H_u$ implies that $\frac{\partial F}{\partial t}(u,t) = (H_t)_u$. Thus it follows from the arbitrariness of (u,t) that f_t $(0 \le t < T)$ is the regularized mean curvature flow for f. This completes the proof. q.e.d.

Assume that the codimension of f is equal to one. Denote by $\widetilde{\mathcal{H}}$ (resp. $\widetilde{\mathcal{V}}$) the horizontal (resp.

vertical) distribution of ϕ. Denote by $\mathrm{pr}_{\widetilde{\mathcal{H}}}$ (resp. $\mathrm{pr}_{\widetilde{\mathcal{V}}}$) the orthogonal projection of TV onto $\widetilde{\mathcal{H}}$ (resp. $\widetilde{\mathcal{V}}$). For simplicity, for $X \in TV$, we denote $\mathrm{pr}_{\widetilde{\mathcal{H}}}(X)$ (resp. $\mathrm{pr}_{\widetilde{\mathcal{V}}}(X)$) by $X_{\widetilde{\mathcal{H}}}$ (resp. $X_{\widetilde{\mathcal{V}}}$). Define a distribution $\mathcal{H}_t$ on M by $f_{t*}((\mathcal{H}_t)_u) = f_{t*}(T_uM) \cap \widetilde{\mathcal{H}}_{f_t(u)}$ $(u \in M)$ and a distribution $\mathcal{V}_t$ on M by $f_{t*}((\mathcal{V}_t)_u) = \widetilde{\mathcal{V}}_{f_t(u)}$ $(u \in M)$. Note that $\mathcal{V}_t$ is independent of the choice of $t \in [0,T)$. Denote by g_t, h_t, A_t, H_t and ξ_t the induced metric, the second fundamental form, the shape tensor and the regularized mean curvature vector and the unit normal vector field of f_t, respectively. The group G acts on M through f_t. Since $\phi : V \to V/G$ is a G-orbibundle and $\widetilde{\mathcal{H}}$ is a connection of the orbibundle, it follows from Proposition 5.1 that this action $G \curvearrowright M$ is independent of the choice of $t \in [0,T)$. It is clear that quantities g_t, h_t, A_t and H_t are G-invariant. Also, let ∇^t be the Riemannian connection of g_t. Let π_M be the projection of $M \times [0,T)$ onto M. For a vector bundle E over M, denote by π_M^*E the induced bundle of E by π_M. Also denote by $\Gamma(E)$ the space of all sections of E. Define a section g of $\pi_M^*(T^{(0,2)}M)$ by $g(u,t) = (g_t)_u$ $((u,t) \in M \times [0,T))$, where $T^{(0,2)}M$ is the $(0,2)$-tensor bundle of M. Similarly, we define a section h of $\pi_M^*(T^{(0,2)}M)$, a section A of $\pi_M^*(T^{(1,1)}M)$, sections H and ξ of the induced bundle F^*TV of TV by F. We regard H and ξ as V-valued functions over $M \times [0,T)$ under the identification of $T_{F(u,t)}V$'s $((u,t) \in M \times [0,T))$ and V. Define a subbundle $\mathcal{H}$ (resp. $\mathcal{V}$) of π_M^*TM by $\mathcal{H}_{(u,t)} := (\mathcal{H}_t)_u$ (resp. $\mathcal{V}_{(u,t)} := (\mathcal{V}_t)_u$). Denote by $\mathrm{pr}_{\mathcal{H}}$ (resp. $\mathrm{pr}_{\mathcal{V}}$) the orthogonal projection of $\pi_M^*(TM)$ onto $\mathcal{H}$ (resp. $\mathcal{V}$). For simplicity, for $X \in \pi_M^*(TM)$, we denote $\mathrm{pr}_{\mathcal{H}}(X)$ (resp. $\mathrm{pr}_{\mathcal{V}}(X)$) by $X_{\mathcal{H}}$ (resp. $X_{\mathcal{V}}$). The bundle $\pi_M^*(TM)$ is regarded as a subbundle of $T(M \times [0,T))$. For a section B of $\pi_M^*(T^{(r,s)}M)$, we define $\dfrac{\partial B}{\partial t}$ by $\left(\dfrac{\partial B}{\partial t}\right)_{(u,t)} := \dfrac{dB_{(u,t)}}{dt}$, where the right-hand side of this relation is the derivative of the vector-valued function $t \mapsto B_{(u,t)}$ $(\in T_u^{(r,s)}M)$. Also, we define a section $B_{\mathcal{H}}$ of $\pi_M^*(T^{(r,s)}M)$ by

$$B_{\mathcal{H}} = (\underbrace{\mathrm{pr}_{\mathcal{H}} \otimes \cdots \otimes \mathrm{pr}_{\mathcal{H}}}_{(r-\text{times})}) \circ B \circ (\underbrace{\mathrm{pr}_{\mathcal{H}} \otimes \cdots \otimes \mathrm{pr}_{\mathcal{H}}}_{(s-\text{times})}).$$

The restriction of $B_{\mathcal{H}}$ to $\mathcal{H} \times \cdots \times \mathcal{H}$ (s-times) is regarded as a section of the (r,s)-tensor bundle $\mathcal{H}^{(r,s)}$ of $\mathcal{H}$. This restriction also is denoted by the same symbol $B_{\mathcal{H}}$. For a tangent vector field X on M (or an open set U of M), we define a section $\bar{X}$ of π_M^*TM (or $\pi_M^*TM|_U$) by $\bar{X}_{(u,t)} := X_u$ $((u,t) \in M \times [0,T))$. Denote by $\widetilde{\nabla}$ the Riemannian connection of V. Define a connection ∇ of π_M^*TM by

$$(\nabla_X Y)_{(\cdot,t)} := \nabla^t_X Y_{(\cdot,t)} \text{ and } \nabla_{\frac{\partial}{\partial t}} Y := \frac{dY_{(u,\cdot)}}{dt}$$

for $X \in T_{(u,t)}(M \times \{t\})$ and $Y \in \Gamma(\pi_M^*TM)$, where $\dfrac{dY_{(u,t)}}{dt}$ is the derivative of the vector-valued function $t \mapsto Y_{(u,t)}$ $(\in T_uM)$. Define a connection $\nabla^{\mathcal{H}}$ of $\mathcal{H}$ by $\nabla^{\mathcal{H}}_X Y := (\nabla_X Y)_{\mathcal{H}}$ for $X \in T(M \times [0,T))$ and $Y \in \Gamma(\mathcal{H})$. Similarly, define a connection $\nabla^{\mathcal{V}}$ of $\mathcal{V}$ by $\nabla^{\mathcal{V}}_X Y := (\nabla_X Y)_{\mathcal{V}}$ for $X \in T(M \times [0,T))$ and $Y \in \Gamma(\mathcal{V})$. Now we shall derive the evolution equations for some geometric quantities. First we derive the following evolution equation for $g_{\mathcal{H}}$.

Lemma 5.2([K3]). *The sections $(g_{\mathcal{H}})_t$'s of $\pi_M^*(T^{(0,2)}M)$ satisfy the following evolution equation:*

$$\frac{\partial g_{\mathcal{H}}}{\partial t} = -2||H||h_{\mathcal{H}},$$

where $||H|| := \sqrt{\langle H, H\rangle}$.

Proof. Take $X, Y \in \Gamma(TM)$. We have

$$
\begin{aligned}
&\frac{\partial g_{\mathcal{H}}}{\partial t}(\bar{X},\bar{Y}) = \frac{\partial}{\partial t} g_{\mathcal{H}}(\bar{X},\bar{Y}) = \frac{\partial}{\partial t} g(\bar{X}_{\mathcal{H}},\bar{Y}_{\mathcal{H}}) = \frac{\partial}{\partial t}\langle F_*\bar{X}_{\mathcal{H}}, F_*\bar{Y}_{\mathcal{H}}\rangle \\
&= \left\langle \frac{\partial}{\partial t}(\bar{X}_{\mathcal{H}}F), \bar{Y}_{\mathcal{H}}F\right\rangle + \left\langle \bar{X}_{\mathcal{H}}F, \frac{\partial}{\partial t}(\bar{Y}_{\mathcal{H}}F)\right\rangle \\
&= \left\langle \bar{X}_{\mathcal{H}}\left(\frac{\partial F}{\partial t}\right) + \left[\frac{\partial}{\partial t}, \bar{X}_{\mathcal{H}}\right] F, \bar{Y}_{\mathcal{H}}F\right\rangle + \left\langle \bar{X}_{\mathcal{H}}F, \bar{Y}_{\mathcal{H}}\left(\frac{\partial F}{\partial t}\right) + \left[\frac{\partial}{\partial t}, \bar{Y}_{\mathcal{H}}\right] F\right\rangle \\
&= \langle \bar{X}_{\mathcal{H}}(||H||\xi), \bar{Y}_{\mathcal{H}}F\rangle + \langle \bar{X}_{\mathcal{H}}F, \bar{Y}_{\mathcal{H}}(||H||\xi)\rangle \\
&= -||H||g(A\bar{X}_{\mathcal{H}},\bar{Y}_{\mathcal{H}}) - ||H||g(\bar{X}_{\mathcal{H}},A\bar{Y}_{\mathcal{H}}) = -2||H||h_{\mathcal{H}}(\bar{X},\bar{Y}),
\end{aligned}
$$

where we use $\left[\frac{\partial}{\partial t}, \bar{X}_{\mathcal{H}}\right] \in \mathcal{V}$ and $\left[\frac{\partial}{\partial t}, \bar{Y}_{\mathcal{H}}\right] \in \mathcal{V}$. Thus we obtain the desired evolution equation.

q.e.d.

Next we derive the following evolution equation for ξ.

Lemma 5.3([K3]). *The unit normal vector fields ξ_t's satisfy the following evolution equation:*

$$\frac{\partial \xi}{\partial t} = -F_*(\mathrm{grad}_g||H||),$$

where $\mathrm{grad}_g(||H||)$ is the element of $\pi_M^*(TM)$ such that $d||H||(X) = g(\mathrm{grad}_g||H||, X)$ for any $X \in \pi_M^*(TM)$.

Proof. Since $\langle \xi, \xi\rangle = 1$, we have $\langle \frac{\partial \xi}{\partial t}, \xi\rangle = 0$. Hence $\frac{\partial \xi}{\partial t}$ is tangent to $f_t(M)$. Take any $(u_0, t_0) \in M \times [0, T)$. Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal base of $T_{u_0}M$ with respect to $g_{(u_0,t_0)}$. By the Fourier expanding $\frac{\partial \xi}{\partial t}|_{t=t_0}$, we have

$$
\begin{aligned}
&\left(\frac{\partial \xi}{\partial t}\right)_{(u_0,t_0)} = \sum \left\langle \left(\frac{\partial \xi}{\partial t}\right)_{(u_0,t_0)}, f_{t_0*}(\bar{e}_i|_{t=t_0})\right\rangle f_{t_0*}(\bar{e}_i|_{t=t_0}) \\
&= -\sum \left\langle \xi_{(u_0,t_0)}, \left.\frac{\partial f_{t*}(\bar{e}_i)}{\partial t}\right|_{t=t_0}\right\rangle f_{t_0*}(\bar{e}_i|_{t=t_0}) = -\sum \left\langle \xi_{(u_0,t_0)}, \left.\frac{\partial}{\partial t}(\bar{e}_iF)\right|_{t=t_0}\right\rangle f_{t_0*}(\bar{e}_i|_{t=t_0}) \\
&= -\sum \left\langle \xi_{(u_0,t_0)}, \bar{e}_i\left(\left.\frac{\partial F}{\partial t}\right|_{t=t_0}\right)\right\rangle f_{t_0*}(\bar{e}_i|_{t=t_0}) = -\sum \langle \xi_{(u_0,t_0)}, (\bar{e}_iH)|_{t=t_0}\rangle f_{t_0*}(\bar{e}_i|_{t=t_0}) \\
&= -\sum (\bar{e}_i||H||)|_{t=t_0} f_{t_0*}(\bar{e}_i|_{t=t_0}) = -\sum g_{(u_0,t_0)}(\mathrm{grad}_{g_{(u_0,t_0)}}||H_{(u_0,t_0)}||, \bar{e}_i|_{t=t_0}) f_{t_0*}(\bar{e}_i|_{t=t_0}) \\
&= -f_{t_0*}(\mathrm{grad}_{g_{(u_0,t_0)}}||H_{(u_0,t_0)}||) = -(F_*(\mathrm{grad}_g||H||))_{(u_0,t_0)},
\end{aligned}
$$

where we use $\left[\frac{\partial}{\partial t}, \bar{e}_i\right] = 0$. Here we note that $\sum(\cdot)_i$ means $\lim_{k\to\infty} \sum_{i\in S_k}(\cdot)_i$ as $S_k := \{i \mid |(\cdot)_i| > \frac{1}{k}\}$ $(k \in \mathbb{N})$. This completes the proof. q.e.d.

Let S_t $(0 \le t < T)$ be a C^∞-family of a (r,s)-tensor fields on M and S a section of $\pi_M^*(T^{(r,s)}M)$ defined by $S_{(u,t)} := (S_t)_u$. We define a section $\triangle_{\mathcal{H}}S$ of $\pi_M^*(T^{(r,s)}M)$ by

$$(\triangle_{\mathcal{H}}S)_{(u,t)} := \sum_{i=1}^{n} \nabla_{e_i}\nabla_{e_i}S,$$

where ∇ is the connection of $\pi_M^*(T^{(r,s)}M)$ (or $\pi_M^*(T^{(r,s+1)}M)$) induced from ∇ and $\{e_1, \cdots, e_n\}$ is an orthonormal base of $\mathcal{H}_{(u,t)}$ with respect to $(g_{\mathcal{H}})_{(u,t)}$. Also, we define a section $\bar{\triangle}_{\mathcal{H}}S_{\mathcal{H}}$ of $\mathcal{H}^{(r,s)}$

by

$$(\bar{\triangle}_{\mathcal{H}}S_{\mathcal{H}})_{(u,t)} := \sum_{i=1}^{n} \nabla^{\mathcal{H}}_{e_i}\nabla^{\mathcal{H}}_{e_i}S_{\mathcal{H}},$$

where $\nabla^{\mathcal{H}}$ is the connection of $\mathcal{H}^{(r,s)}$ (or $\mathcal{H}^{(r,s+1)}$) induced from $\nabla^{\mathcal{H}}$ and $\{e_1, \cdots, e_n\}$ is as above. Let $\mathcal{A}^{\phi}$ be the section of $T^*V \otimes T^*V \otimes TV$ defined by

$$\mathcal{A}^{\phi}_X Y := (\widetilde{\nabla}_{X_{\widetilde{\mathcal{H}}}} Y_{\widetilde{\mathcal{H}}})_{\widetilde{\mathcal{V}}} + (\widetilde{\nabla}_{X_{\widetilde{\mathcal{H}}}} Y_{\widetilde{\mathcal{V}}})_{\widetilde{\mathcal{H}}} \quad (X, Y \in TV).$$

Also, let $\mathcal{T}^{\phi}$ be the section of $T^*V \otimes T^*V \otimes TV$ defined by

$$\mathcal{T}^{\phi}_X Y := (\widetilde{\nabla}_{X_{\widetilde{\mathcal{V}}}} Y_{\widetilde{\mathcal{H}}})_{\widetilde{\mathcal{V}}} + (\widetilde{\nabla}_{X_{\widetilde{\mathcal{V}}}} Y_{\widetilde{\mathcal{V}}})_{\widetilde{\mathcal{H}}} \quad (X, Y \in TV).$$

Also, let $\mathcal{A}_t$ be the section of $T^*M \otimes T^*M \otimes TM$ defined by

$$(\mathcal{A}_t)_X Y := (\nabla^t_{X_{\mathcal{H}_t}} Y_{\mathcal{H}_t})_{\mathcal{V}_t} + (\nabla^t_{X_{\mathcal{H}_t}} Y_{\mathcal{V}_t})_{\mathcal{H}_t} \quad (X, Y \in TM).$$

Also let $\mathcal{A}$ be the section of $\pi_M^*(T^*M \otimes T^*M \otimes TM)$ defined in terms of $\mathcal{A}_t$'s ($t \in [0,T)$). Also, let $\mathcal{T}_t$ be the section of $T^*M \otimes T^*M \otimes TM$ defined by

$$(\mathcal{T}_t)_X Y := (\nabla^t_{X_{\mathcal{V}_t}} Y_{\mathcal{V}_t})_{\mathcal{H}_t} + (\nabla^t_{X_{\mathcal{V}_t}} Y_{\mathcal{H}_t})_{\mathcal{V}_t} \quad (X, Y \in TM).$$

Also let $\mathcal{T}$ be the section of $\pi_M^*(T^*M \otimes T^*M \otimes TM)$ defined in terms of $\mathcal{T}_t$'s ($t \in [0,T)$). Clearly we have

$$F_*(\mathcal{A}_X Y) = \mathcal{A}^{\phi}_{F_*X} F_*Y$$

for $X, Y \in \mathcal{H}$ and

$$F_*(\mathcal{T}_W X) = \mathcal{T}^{\phi}_{F_*W} F_*X$$

for $X \in \mathcal{H}$ and $W \in \mathcal{V}$. Let E be a vector bundle over M. For a section S of $\pi_M^*(T^{(0,r)}M \otimes E)$, we define $\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}} S(\cdots, \overset{j}{\bullet}, \cdots, \overset{k}{\bullet}, \cdots)$ by

$$(\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}} S(\cdots, \overset{j}{\bullet}, \cdots, \overset{k}{\bullet}, \cdots))_{(u,t)} = \sum_{i=1}^{n} S_{(u,t)}(\cdots, \overset{j}{e_i}, \cdots, \overset{k}{e_i}, \cdots)$$

($(u,t) \in M \times [0,T)$), where $\{e_1, \cdots, e_n\}$ is an orthonormal base of $\mathcal{H}_{(u,t)}$ with respect to $(g_{\mathcal{H}})_{(u,t)}$, $S(\cdots, \overset{j}{\bullet}, \cdots, \overset{k}{\bullet}, \cdots)$ means that $\bullet$ is entried into the j-th component and the k-th component of S and $S_{(u,t)}(\cdots, \overset{j}{e_i}, \cdots, \overset{k}{e_i}, \cdots)$ means that e_i is entried into the j-th component and the k-th component of $S_{(u,t)}$.

Then we have the following relation.

Lemma 5.4([K3]). *Let S be a section of $\pi_M^*(T^{(0,2)}M)$ which is symmetric with respect to g. Then we have*

$$\begin{aligned}(\triangle_{\mathcal{H}}S)_{\mathcal{H}}(X,Y) = & (\triangle^{\mathcal{H}}_{\mathcal{H}}S_{\mathcal{H}})(X,Y)\\ & -2\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}((\nabla_{\bullet}S)(\mathcal{A}_{\bullet}X,Y)) - 2\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}((\nabla_{\bullet}S)(\mathcal{A}_{\bullet}Y,X))\\ & -\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}S(\mathcal{A}_{\bullet}(\mathcal{A}_{\bullet}X),Y) - \mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}S(\mathcal{A}_{\bullet}(\mathcal{A}_{\bullet}Y),X)\\ & -\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}S((\nabla_{\bullet}\mathcal{A})_{\bullet}X,Y) - \mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}S((\nabla_{\bullet}\mathcal{A})_{\bullet}Y,X)\\ & -2\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}S(\mathcal{A}_{\bullet}X,\mathcal{A}_{\bullet}Y)\end{aligned}$$

for $X, Y \in \mathcal{H}$, where ∇ is the connection of $\pi_M^*(T^{(1,2)}M)$ induced from ∇.

Proof. Take any $(u_0, t_0) \in M \times [0, T)$. Let $\{e_1, \cdots, e_n\}$ be an orthonormal base of $\mathcal{H}_{(u_0, t_0)}$ with respect to $(g_{\mathcal{H}})_{(u_0, t_0)}$. Take any $X, Y \in \mathcal{H}_{(u_0, t_0)}$. Let $\widetilde{X}$ be a section of $\mathcal{H}$ on a neighborhood of (u_0, t_0) with $\widetilde{X}_{(u_0, t_0)} = X$ and $(\nabla^{\mathcal{H}} X)_{(u_0, t_0)} = 0$. Similarly we define $\widetilde{Y}$ and $\widetilde{e}_i$. Let $W = X, Y$ or e_i. Then, it follows from $(\nabla^{\mathcal{H}}_{e_i} \widetilde{W})_{(u_0, t_0)} = 0$, $(\nabla_{e_i} \widetilde{W})_{(u_0, t_0)} = \mathcal{A}_{e_i} W$ and the skew-symmetricness of $\mathcal{A}|_{\mathcal{H} \times \mathcal{H}}$ that

$$
\begin{aligned}
(\triangle_{\mathcal{H}} S)_{\mathcal{H}}(X, Y) &= \sum_{i=1}^{n} (\nabla_{e_i} \nabla_{e_i} S)(X, Y) \\
= \sum_{i=1}^{n} (\nabla^{\mathcal{H}}_{e_i} \nabla^{\mathcal{H}}_{e_i} S_{\mathcal{H}})(X, Y) &- 2 \sum_{i=1}^{n} ((\nabla_{e_i} S)(\mathcal{A}_{e_i} X, Y) + (\nabla_{e_i} S)(\mathcal{A}_{e_i} Y, X)) \\
- \sum_{i=1}^{n} (S(\mathcal{A}_{e_i}(\mathcal{A}_{e_i} X), Y) &+ S(\mathcal{A}_{e_i}(\mathcal{A}_{e_i} Y), X)) - 2 \sum_{i=1}^{n} S(\mathcal{A}_{e_i} X, \mathcal{A}_{e_i} Y) \\
- \sum_{i=1}^{n} (S((\nabla_{e_i} \mathcal{A})_{e_i} X, Y) &+ S((\nabla_{e_i} \mathcal{A})_{e_i} Y, X)).
\end{aligned}
$$

The right-hand side of this relation is equal to the right-hand side of the relation in the statement. This completes the proof. q.e.d.

Also we have the following Simons-type identity.

Lemma 5.5([K3]). *We have*

$$\triangle_{\mathcal{H}} h = \nabla d ||H|| + ||H|| (A^2)_\sharp - (\mathrm{Tr}\, (A^2)_{\mathcal{H}}) h,$$

where $(A^2)_\sharp$ is the element of $\Gamma(\pi_M^* T^{(0,2)} M)$ defined by $(A^2)_\sharp(X, Y) := g(A^2 X, Y)$ $(X, Y \in \pi_M^* TM)$.

Proof. Take $X, Y, Z, W \in \pi_M^*(TM)$. Since the ambient space V is flat, it follows from the Ricci's identity, the Gauss equation and the Codazzi equation that

$$
\begin{aligned}
(\nabla_X \nabla_Y h)(Z, W) - (\nabla_Z \nabla_W h)(X, Y) &= (\nabla_X \nabla_Z h)(Y, W) - (\nabla_Z \nabla_X h)(Y, W) \\
= h(X, Y) h(AZ, W) - h(Z, Y) h(AX, W) &+ h(X, W) h(AZ, Y) - h(Z, W) h(AX, Y).
\end{aligned}
$$

By using this relation, we obtain the desired relation. q.e.d.

Note. In the sequel, we omit the notation F_* for simplicity.

Define a section $\mathcal{R}$ of $\pi_M^*(\mathcal{H}^{(0,2)})$ by

$$
\begin{aligned}
\mathcal{R}(X, Y) := &\ \mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}} h(\mathcal{A}_\bullet(\mathcal{A}_\bullet X), Y) + \mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}} h(\mathcal{A}_\bullet(\mathcal{A}_\bullet Y), X) \\
&+ \mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}} h((\nabla_\bullet \mathcal{A})_\bullet X, Y) + \mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}} h((\nabla_\bullet \mathcal{A})_\bullet Y, X) \\
&+ 2\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}} (\nabla_\bullet h)(\mathcal{A}_\bullet X, Y) + 2\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}} (\nabla_\bullet h)(\mathcal{A}_\bullet Y, X) \\
&+ 2\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}} h(\mathcal{A}_\bullet X, \mathcal{A}_\bullet Y) \qquad (X, Y \in \mathcal{H}).
\end{aligned}
$$

From Lemmas 5.3, 5.4 and 5.5, we derive the following evolution equation for $(h_{\mathcal{H}})_t$s.

Theorem 5.6([K3]). *The sections $(h_{\mathcal{H}})_t$'s of $\pi_M^*(T^{(0,2)}M)$ satisfies the following evolution equation:*

$$\frac{\partial h_{\mathcal{H}}}{\partial t}(X,Y) = (\triangle_{\mathcal{H}}^{\mathcal{H}} h_{\mathcal{H}})(X,Y) - 2||H||((A_{\mathcal{H}})^2)_\sharp(X,Y) - 2||H||((\mathcal{A}_\xi^\phi)^2)_\sharp(X,Y)$$
$$+\mathrm{Tr}\left((A_{\mathcal{H}})^2 - ((\mathcal{A}_\xi^\phi)^2)_{\mathcal{H}}\right) h_{\mathcal{H}}(X,Y) - \mathcal{R}(X,Y)$$

for $X,Y \in \mathcal{H}$.

Proof. Take $X,Y \in \mathcal{H}_{(u,t)}$. Easily we have

$$AX = A_{\mathcal{H}}X + \mathcal{A}_\xi^\phi X \quad \text{and} \quad (A^2)_{\mathcal{H}}X = (A_{\mathcal{H}})^2X - (\mathcal{A}_\xi^\phi)^2X, \tag{5.1}$$

where we use

$$\left(\widetilde{\nabla}_W\xi\right)_{\widetilde{\mathcal{H}}} = \left(\widetilde{\nabla}_\xi W + [W,\xi]\right)_{\widetilde{\mathcal{H}}} = \left(\widetilde{\nabla}_\xi W\right)_{\widetilde{\mathcal{H}}} = \mathcal{A}_\xi W$$

for $W \in \Gamma(\widetilde{\mathcal{V}})$ because of $[W,\xi] \in \Gamma(\widetilde{\mathcal{V}})$. Also, since $\left[\frac{\partial}{\partial t}, \bar{X}_{\mathcal{H}}\right] \in \mathcal{V}$, we have

$$\left[\frac{\partial}{\partial t}, \bar{X}_{\mathcal{H}}\right] = 2||H||\mathcal{A}_\xi^\phi \bar{X}_{\mathcal{H}}. \tag{5.2}$$

From Lemma 5.3, (5.1) and (5.2), we have

$$\begin{aligned}
&\frac{\partial h_{\mathcal{H}}}{\partial t}(X,Y) = \frac{\partial}{\partial t}(h_{\mathcal{H}}(\bar{X},\bar{Y})) = \frac{\partial}{\partial t}\langle\xi, \bar{X}_{\mathcal{H}}(\bar{Y}_{\mathcal{H}}F)\rangle \\
&= \langle\frac{\partial\xi}{\partial t}, \bar{X}_{\mathcal{H}}(\bar{Y}_{\mathcal{H}}F)\rangle + \langle\xi, \frac{\partial}{\partial t}\left(\bar{X}_{\mathcal{H}}(\bar{Y}_{\mathcal{H}}F)\right)\rangle \\
&= -\langle F_*(\mathrm{grad}_g||H||), \widetilde{\nabla}_X F_*\bar{Y}_{\mathcal{H}}\rangle + \langle\xi, X\left(\bar{Y}_{\mathcal{H}}\left(\frac{\partial F}{\partial t}\right)\right)\rangle \\
&\quad +\langle\xi, X([\frac{\partial}{\partial t}, \bar{Y}_{\mathcal{H}}]F)\rangle + \langle\xi, [\frac{\partial}{\partial t}, \bar{X}_{\mathcal{H}}](\bar{Y}_{\mathcal{H}}F)\rangle \\
&= -g(\mathrm{grad}_g||H||, \nabla_X\bar{Y}_{\mathcal{H}}) + X(\bar{Y}_{\mathcal{H}}||H||) - ||H||\langle\xi, \widetilde{\nabla}_X F_*(A(\bar{Y}_{\mathcal{H}})\rangle \\
&\quad +\langle\xi, \widetilde{\nabla}_X F_*([\frac{\partial}{\partial t}, \bar{Y}_{\mathcal{H}}])\rangle + \langle\xi, \widetilde{\nabla}_{[\frac{\partial}{\partial t}, \bar{X}_{\mathcal{H}}]}F_*\bar{Y}_{\mathcal{H}}\rangle \\
&= (\nabla d||H||)(X,Y) - ||H||h_{\mathcal{H}}(X, A_{\mathcal{H}}Y) + ||H||h(X, \mathcal{A}_\xi^\phi Y) + 2||H||h(\mathcal{A}_\xi^\phi X, Y) \\
&= (\nabla d||H||)(X,Y) - ||H||g_{\mathcal{H}}((A_{\mathcal{H}})^2X, Y) - 3||H||g((\mathcal{A}_\xi^\phi)^2X, Y)
\end{aligned}$$

From this relation and the Simons-type identity in Lemma 5.5, we have

$$\begin{aligned}
\frac{\partial h_{\mathcal{H}}}{\partial t} &= \triangle_{\mathcal{H}}h - 2||H||((A_{\mathcal{H}})^2)_\sharp - 2||H||((\mathcal{A}_\xi^\phi)^2)_\sharp \\
&\quad +\mathrm{Tr}\left((A_{\mathcal{H}})^2 - ((\mathcal{A}_\xi^\phi)^2)_{\mathcal{H}}\right) h_{\mathcal{H}}.
\end{aligned} \tag{5.3}$$

Substituting the relation in Lemma 5.4 into (5.3), we obtain the desired relation. q.e.d.

For $\mathcal{R}$, we can show the following fact.

Lemma 5.7([K3]). *For $X \in \mathcal{H}$, we have*

$$\begin{aligned}
\mathcal{R}(X,X) &= 4\mathrm{Tr}^\bullet_{g_{\mathcal{H}}}\langle\mathcal{A}_\bullet^\phi X, \mathcal{A}_\bullet^\phi(A_{\mathcal{H}}X)\rangle + 4\mathrm{Tr}^\bullet_{g_{\mathcal{H}}}\langle\mathcal{A}_\bullet^\phi X, \mathcal{A}_X^\phi(A_{\mathcal{H}}\bullet)\rangle \\
&\quad +3\mathrm{Tr}^\bullet_{g_{\mathcal{H}}}\langle(\widetilde{\nabla}_\bullet\mathcal{A}^\phi)_\xi X, \mathcal{A}_\bullet^\phi X\rangle + 2\mathrm{Tr}^\bullet_{g_{\mathcal{H}}}\langle(\widetilde{\nabla}_\bullet\mathcal{A}^\phi)_\bullet X, \mathcal{A}_\xi^\phi X\rangle \\
&\quad +\mathrm{Tr}^\bullet_{g_{\mathcal{H}}}\langle\mathcal{A}_\bullet^\phi X, (\widetilde{\nabla}_X\mathcal{A}^\phi)_\xi\bullet\rangle
\end{aligned}$$

and hence $\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}\mathcal{R}(\bullet,\bullet)=0$.

By using Theorem 5.6 and Lemma 5.7, we can show the following evolution equation for $||H_t||$'s.

Corollary 5.8([K3]). *The norms* $||H_t||$*'s of* H_t *satisfy the following evolution equation:*

$$\frac{\partial||H||}{\partial t}=\triangle_{\mathcal{H}}||H||+||H||\mathrm{Tr}(A_{\mathcal{H}})^2-3||H||\mathrm{Tr}((\mathcal{A}^{\phi}_{\xi})^2)_{\mathcal{H}}.$$

Remark 5.1. From the evolution equations obtained in this section, the evolution equations for the corresponding geometric quantities of $\overline{f}_t(:\overline{M}\hookrightarrow V/G)$ are derived, respectively. In the case where the G-action is free and hence V/G is a (complete) Riemannian manifold, the above derived evolution equations coincide with the evolution equations for the corresponding geometric quantities along the mean curvature flow in a complete Riemannian manifold which were given by Huisken [Hu2]. That is, the discussion in this section give a new proof of the evolution equations in [Hu2] in the case where the ambient complete Riemannian manifold occurs as V/G. In the proof of [Hu2], one need to take local coordinates of the ambient space to derive the evolution equations. On the other hand, in our proof, one need not take local coordinates of the ambient space V and can identify the tangent space of the ambient space V at each point with V. These are an advantage of our proof.

6 Horizontally strongly convexity preservability theorem

Let $G\curvearrowright V$ be an isometric almost free action with minimal regularizable orbit of a Hilbert Lie group G on a Hilbert space V equipped with an inner product $\langle\ ,\ \rangle$ and $\phi:V\to V/G$ the orbit map. Denote by $\widetilde{\nabla}$ the Riemannian connection of V. Set $n:=\dim V/G\ -1$. Let $M(\subset V)$ be a G-invariant hypersurface in V such that $\phi(M)$ is compact. Let f be an inclusion map of M into V and f_t $(0\leq t<T)$ the regularized mean curvature flow for f. We use the notations in Section 5. In the sequel, we omit the notation f_{t*} for simplicity. Set

$$L:=\max_{(X_1,\cdots,X_5)\in\widetilde{\mathcal{H}}_1^5}|\langle\mathcal{A}^{\phi}_{X_1}((\widetilde{\nabla}_{X_2}\mathcal{A}^{\phi})_{X_3}X_4),\,X_5\rangle|,$$

where $\widetilde{\mathcal{H}}_1:=\{X\in\widetilde{\mathcal{H}}\,|\ ||X||=1\}$. Assume that $L<\infty$. Note that $L<\infty$ in the case where V/G is compact. Then we obtain the following horizontally strongly convexity preservability theorem by using the evolution equations in Section 5 and the maximum principle for C^{∞}-family of G-invariant symmetric $(0,2)$-tensor fields on M (see [K3]).

Theorem 6.1([K3]). *If* M *satisfies* $||H_0||^2(h_{\mathcal{H}})_{(\cdot,0)}>2n^2L(g_{\mathcal{H}})_{(\cdot,0)}$*, then* $T<\infty$ *holds and* $||H_t||^2(h_{\mathcal{H}})_{(\cdot,t)}>2n^2L(g_{\mathcal{H}})_{(\cdot,t)}$ *holds for all* $t\in[0,T)$.

7 Strongly convex preservability theorem in the orbit space

Let V, G and ϕ be as in the previous section. Set $N:=V/G$ and $n:=\dim V/G-1$. Denote by g_N and R_N the Riemannian orbimetric and the curvature orbitensor of N. Also, ∇^N the Riemannian connection of $g_N|_{N\setminus \mathrm{Sing}(N)}$. Since the Riemannian manifold $(N \setminus \mathrm{Sing}(N), g_N|_{N\setminus \mathrm{Sing}(N)})$ is locally homogeneous, the norm $||\nabla^N R_N||$ of $\nabla^N R_N$ (with respect to g_N) is constant over $N \setminus \mathrm{Sing}(N)$. Set $L_N := ||\nabla^N R_N||$. Assume that $L_N < \infty$. Let $\overline{M}$ be a compact suborbifold of codimension one in N immersed by $\overline{f}$ and $\overline{f}_t$ $(t \in [0,T))$ the mean curvature flow for $\overline{f}$. Denote by $\overline{g}_t, \overline{h}_t, \overline{A}_t$ and $\overline{H}_t$ be the induced orbimetric, the second fundamental orbiform, the shape orbitensor and the mean curvature orbifunction of $\overline{f}_t$, respectively, and $\overline{\xi}_t$ the unit normal vector field of $\overline{f}_t|_{\overline{M}\setminus \mathrm{Sing}(\overline{M})}$.

From Theorem 6.1, we obtain the following strongly convexity preservability theorem for compact suborbifolds in N.

Theorem 7.1([K3]). *If $\overline{f}$ satisfies $||\overline{H}_0||^2\overline{h}_0 > n^2 L_N \overline{g}_0$, then $T < \infty$ holds and $||\overline{H}_t||^2\overline{h}_t > n^2 L_N \overline{g}_t$ holds for all $t \in [0,T)$.*

Proof. Set $M := \{(x,u) \in \overline{M} \times V \mid \overline{f}(x) = \phi(u)\}$ and define $f : M \to V$ by $f(x,u) = u$ $((x,u) \in M)$. It is clear that f is an immersion. Denote by H_0 the regularized mean curvature vector of f. Define a curve $c_x : [0,T) \to N$ by $c_x(t) := \overline{f}_t(x)$ $(t \in [0,T))$ and let $(c_x)^L_u$ be the horizontal lift of c_x for u, where $u \in \phi^{-1}(f(x))$. Define an immersion $f_t : M \hookrightarrow V$ by $f_t(x,u) := (c_x)^L_u(t)$ $((x,u) \in M)$. Then f_t $(t \in [0,T))$ is the regularized mean curvature flow for f (see the proof of Proposition 5.1). Denote by g_t, h_t, A^t and H_t the induced metric, the second fundamental form, the shape tensor and the mean curvature vector of f_t, respectively. By the assumption, $\overline{f}_0$ satisfies $||\overline{H}_0||^2\overline{h}_0 > n^2 L_N \overline{g}_0$. Also, we can show $L_N = 2L$ by long calculation, where L is as in the prevoius section. From these facts, we can show that f_0 satisfies $||H_0||^2(h_{\mathcal{H}})_0 > 2n^2 L(g_{\mathcal{H}})_0$. Hence, it follows from Theorem 6.1 that f_t $(t \in [0,T))$ satisfies $||H_t||^2(h_{\mathcal{H}})_t > 2n^2 L(g_{\mathcal{H}})_t$. Furthermore, it follows from this fact that $\overline{f}_t$ $(t \in [0,T))$ satisfies $||\overline{H}_t||^2\overline{h}_t > n^2 L_N \overline{g}_t$. q.e.d.

Remark 7.1. In the case where the G-action is free and hence N is a (complete) Riemannian manifold, Theorem 6.1 implies the strongly convexity preservability theorem by G. Huisken (see [Hu2, Theorem 4.2]).

Acknowledgement

The author is grateful to JSPS for support (Grant-in-Aid for Science Research (C), no.25400076).

References

[AK] A. Adem and M. Klaus, Lectures on orbifolds and group cohomology, Advanced Lectures in Mathematics **16**, Transformation groups and Moduli Spaces of Curves (2010), 1-17.

[BB] J. E. Borzelliono and V. Brunsden, Orbifold homeomorphism and Diffeomorphism groups, Report in International Conference on Infinite Dimensional Lie Groups in Geometry and Representation Theory at Howard University, Washington DC, 2000.

[Ge] C. Gerhart, Curvature Problems, Series in Geometry and Topology, **39**, International Press 2006.

[GKP] P. Gilkey, H. J. Kim and J. H. Park, Eigenforms of the Laplacian for Riemannian V-submersions, Tohoku Math. J. **57** (2005) 505-519.

[Ha] R. S. Hamilton, Three-manifolds with positive Ricci curvature, J. Differential Geom.

17 (1982) 255-306.

[Hu1] G. Huisken, Flow by mean curvature of convex surfaces into spheres, J. Differential Geom. **20** (1984) 237-266.

[Hu2] G. Huisken, Contracting convex hypersurfaces in Riemannian manifolds by their mean curvature, Invent. math. **84** (1986) 463-480.

[HLO] E. Heintze, X. Liu and C. Olmos, Isoparametric submanifolds and a Chevalley-type restriction theorem, Integrable systems, geometry, and topology, 151-190, AMS/IP Stud. Adv. Math. 36, Amer. Math. Soc., Providence, RI, 2006.

[HP] G. Huisken and A. Polden, Geometric evolution equations for hypersurfaces, Calculus of Variations and Geometric Evolution Problems, CIME Lectures at Cetraro of 1966 (S. Hildebrandt and M. Struwe, eds.) Springer, 1999.

[K1] N. Koike, On proper Fredholm submanifolds in a Hilbert space arising from submanifolds in a symmetric space, Japan. J. Math. **28** (2002) 61-80.

[K2] N. Koike, Collapse of the mean curvature flow for equifocal submanifolds, Asian J. Math. **15** (2011) 101-128.

[K3] N. Koike, The mean curvature flow for invariant hypersurfaces in a Hilbert space with an almost free group action, arXiv:math.DG/1210.2539v2.

[O'N] B. O'Neill, The fundamental equations of a submersion, Michigan Math. J. **13** (1966) 459-469.

[P] R. S. Palais, Morse theory on Hilbert manifolds, Topology **2** (1963) 299-340.

[PT] R. S. Palais and C. L. Terng, Critical point theory and submanifold geometry, Lecture Notes in Math. **1353**, Springer, Berlin, 1988.

[Sa] I. Satake, The Gauss-Bonnet theorem fro V-manifolds, J. Math. Soc. Japan **9** (1957) 464-492.

[Sh] T. Shioya, Eigenvalues and suspension structure of compact Riemannian orbifolds with positive Ricci curvature, Manuscripta Math. **99** (1999) 509-516.

[Th] W. P. Thurston, Three-dimensional geometry and topology, Princeton Mathematical Series 35, Princeton University Press, Princeon, N. J., 1997.

[Te] C. L. Terng, Proper Fredholm submanifolds of Hilbert space, J. Differential Geometry **29** (1989) 9-47.

[TT] C.L. Terng and G. Thorbergsson, Submanifold geometry in symmetric spaces, J. Differential Geometry **42** (1995) 665-718.

[Z] X. P. Zhu, Lectures on mean curvature flows, Studies in Advanced Math., AMS/IP, 2002.

Department of Mathematics, Faculty of Science
Tokyo University of Science
Tokyo 162-8601
Japan
E-mail address: koike@ma.kagu.tus.ac.jp

東京理科大学・理学部・数学科　小池　直之